- V-polytopes $P := \operatorname{conv} V$
- H-polytopes $P := \bigcap \mathcal{H}$
- Thm H-polytope $\longrightarrow$ V-polytope

17/10/2022

2.2. Polar duality

- polytopes come in pairs

Def: $P \subset \mathbb{R}^d$, the (polar) dual is the set

$$P^\circ := \{ y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for all } x \in P \}$$

$$= \bigcap_{x \in P} H(x, 1)$$

$\{ y \mid \langle y, x \rangle \leq 1 \}$

Lem: if $P = \operatorname{conv}\{x_1, \dots, x_n\}$

$$\longrightarrow P^\circ := \underbrace{\{ y \in \mathbb{R}^d \mid \langle x_i, y \rangle \leq 1 \text{ for all } i \in [n] \}}_{=: Q}$$

$\longrightarrow P^\circ$ is a polyhedron

Proof:

- $P^\circ \subseteq Q$: trivial
- $Q \subseteq P^\circ$:
  - fix $y \in Q \rightarrow \langle y, x_i \rangle \leq 1 \quad \forall i \in [n]$
  - fix $x \in P$, we need to show that $\langle y, x \rangle \leq 1$

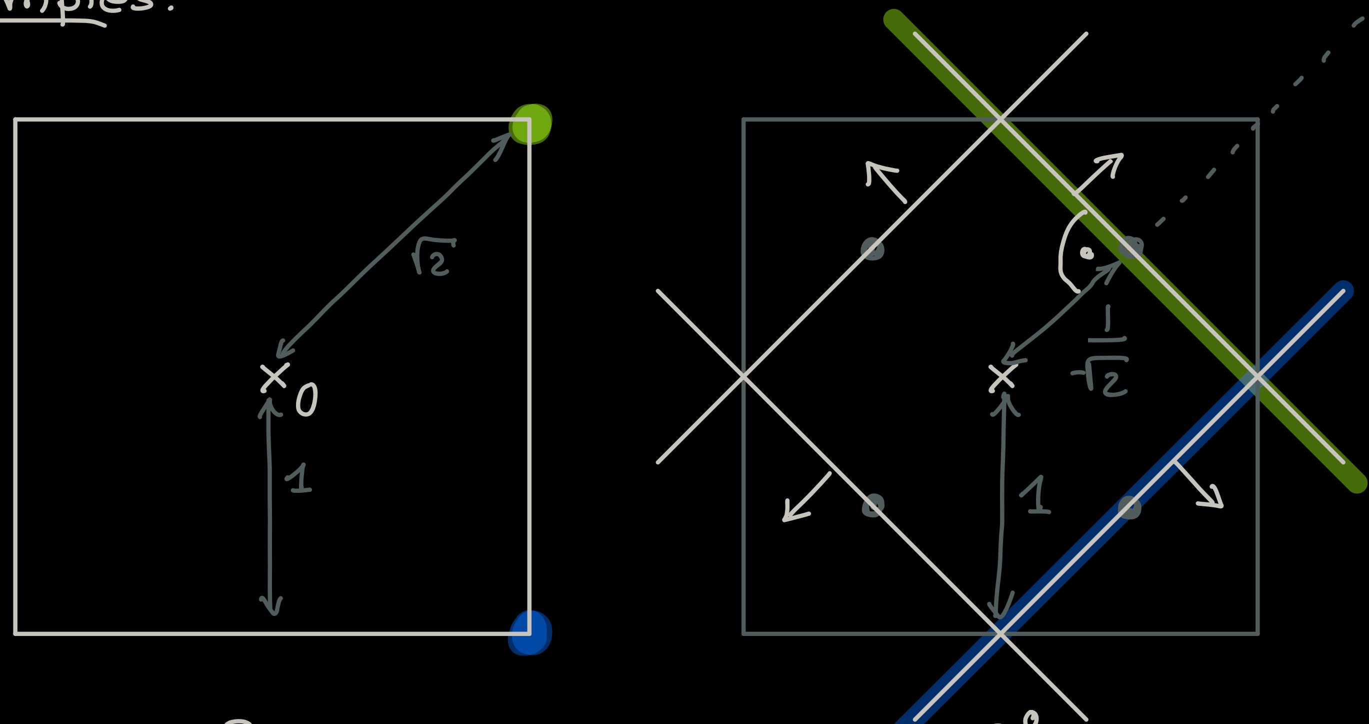
- write $x = \sum_i \alpha_i x_i$ with $\alpha \in \Delta_n$

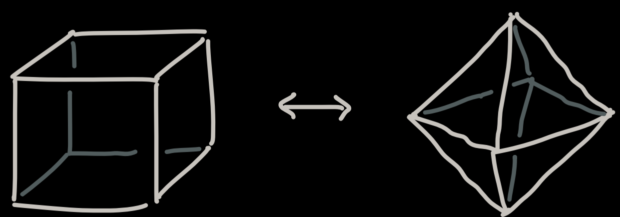
$$\to \langle y, x\rangle = \sum_i \alpha_i \underbrace{\langle y, x_i\rangle}_{\le 1} \le \sum_i \alpha_i = 1$$

$$\to y \in P^\circ$$

$\square$

Examples:





- polar dual P° changes when P is scaled or translated!

- a polytope is self-polar if it is isometric to its polar dual

"of the same shape and size, maybe rotated or reflected"

Open: Which self-polar polytope has the smaller volume? Is it the simplex?

(related to Mahler conjecture)

Lem: if P is a V-polytope with $0 \in \operatorname{int}(P)$
then $P^{\circ\circ} = P$ $\longleftarrow$ "dual" is justified

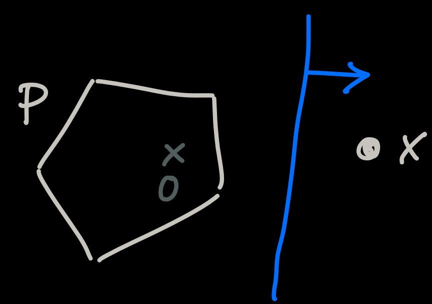
Proof:

$$x \in P^{\circ\circ} \Leftrightarrow \forall y \in P^{\circ}: \langle x, y \rangle \leq 1$$
$$\Leftrightarrow \forall y \in \mathbb{R}^d: \quad y \in P^{\circ} \longrightarrow \langle x, y \rangle \leq 1$$
$$\Leftrightarrow \forall y \in \mathbb{R}^d: \left(\forall x' \in P: \langle x', y \rangle \leq 1\right) \longrightarrow \langle x, y \rangle \leq 1$$

- $P \subseteq P^{\circ\circ}$:
  - fix $x \in P$
  - for all $y \in P$: if $\langle x', y \rangle \leq 1$ for all $x' \in P$ then also when $x' = x$

    $\longrightarrow \langle x, y \rangle \leq 1$ $\Big\}$ $x \in P^{\circ\circ}$

- $P^{\circ\circ} \subseteq P$:



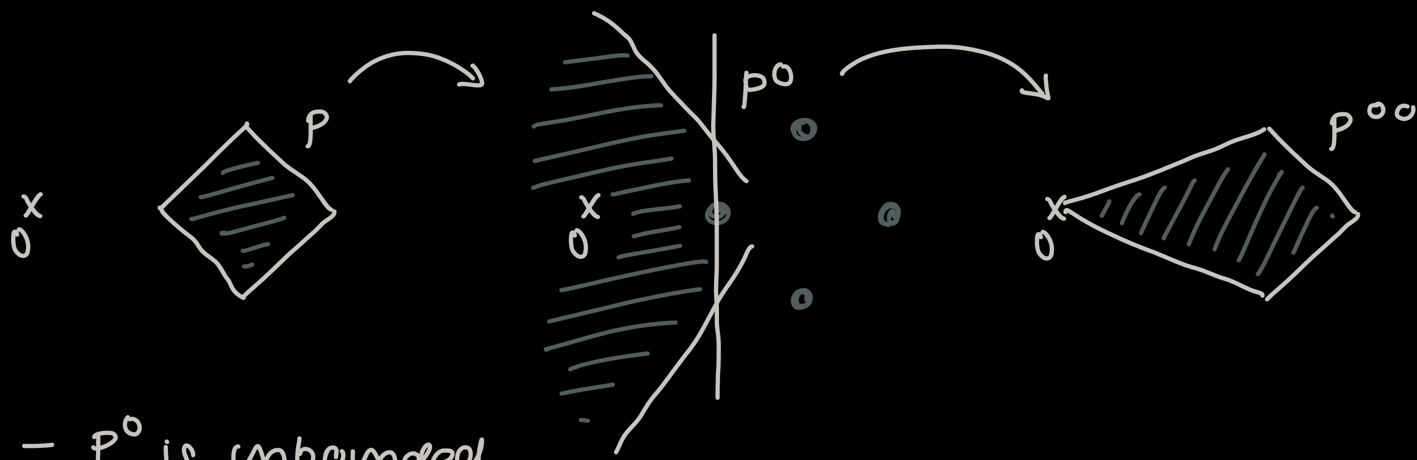
  - suppose $x \notin P$
  - by hyperplane separation theorem exist a hyperplane that separates x from P:

    $$\exists a \in \mathbb{R}^d \setminus \{0\}, b \in \mathbb{R}: \quad \langle x, a/b \rangle > b/b$$
    $$\langle x', a/b \rangle \leq b/b \quad \forall x' \in P$$
  - since $0 \in \operatorname{int}(P)$ we have $b \neq 0$
  - set $y := a/b$

$\longrightarrow \langle x, y \rangle > 1$ while $\langle x', y \rangle \leq 1 \quad \forall x' \in P$

$\longrightarrow x \notin P^{\circ\circ}$ $\square$

<u>Note:</u> if $0 \notin \operatorname{int} P$ then $P^{\circ\circ} \neq P$



- P° is unbounded
- $P^{\circ\circ} = \operatorname{conv}(P \cup \{0\})$

<u>Lem:</u> if $0 \in \operatorname{int}(P)$, then P° is unbounded.

<u>Proof:</u>

- $\exists \varepsilon > 0 : B_\varepsilon(0) \subset P$
- if P° were unbounded, then $\exists\, y_1, y_2, \ldots \in P^\circ$ with $\|y_i\| \longrightarrow \infty$
- $x_i := \varepsilon \cdot \frac{y_i}{\|y_i\|} \longrightarrow \|x_i\| = \varepsilon \longrightarrow x_i \in B_\varepsilon(0) \subset P$
- since $\left.\begin{matrix} y_i \in P^\circ \\ x_i \in P \end{matrix}\right\}$ $1 \geq \langle x_i, y_i \rangle$

$$= \left\langle \varepsilon \frac{y_i}{\|y_i\|}, y_i \right\rangle = \varepsilon \frac{\langle y_i, y_i \rangle}{\|y_i\|}$$

$$= \varepsilon \frac{\|y_i\|^2}{\|y_i\|} = \varepsilon \|y_i\| \longrightarrow \infty \quad \lightning$$

$\square$

<u>Lem:</u> if P is bounded, then $0 \in \operatorname{int}(P^\circ)$

<u>Proof:</u> Ex (not hard) $\square$

Conclusions: (*)

if P is a (bounded) V-polytope with $0 \in \mathrm{int}(P)$
then P° is a (bounded) $\mathcal{H}$-polytope with $0 \in \mathrm{int}(P^\circ)$

$$\mathrm{conv}\{x_1, \dots, x_n\} \longleftrightarrow \bigcap_i H(x_i, 1)$$

Thm: If P is a V-polytope, then P is an $\mathcal{H}$-polytope

Proof

- w.l.o.g. P is full-dimensional
- after translation we can assume $0 \in \mathrm{int}(P)$
- by (*) P° is an $\mathcal{H}$-polytope with $0 \in \mathrm{int}(P^\circ)$
  $\longrightarrow$ P° is also a V-polytope with $0 \in \mathrm{int}(P^\circ)$
  $\longrightarrow$ $P^{\circ\circ}$ is an $\mathcal{H}$-polytope

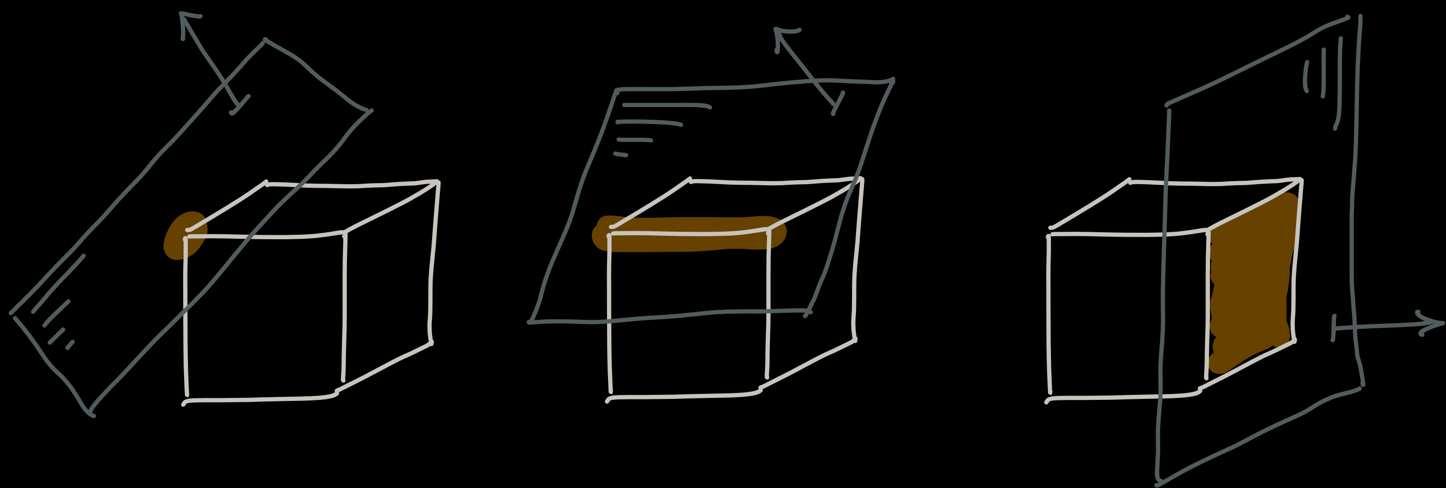
$$P^{\circ\circ} = P$$

$\square$

Finally: polytope $:=$ V-polytope $=$ $\mathcal{H}$-polytope

2.3. Faces

"a face is an intersection with a touching hyperplane"



- But this does not capture everything that we want to call "face"

  missing: P and $\emptyset$

Def:
- $\langle a,x \rangle \le b$ is feasible for P if valid for all $x \in P$
- a face is

$$f := \{ x \in P \mid \langle a,x \rangle = b \} \subseteq P$$

There are three cases: "face-defining hyperplane"

i) $a \neq 0$: $\partial H(a,b)$ is exactly this intuitive "touching hyperplane"

ii) $a=0,\ b=0$: $\langle 0,x \rangle \le 0$ valid for all $x \in P$

$$\longrightarrow \{ x \in P \mid \langle 0,x \rangle = 0 \} = P \text{ is a face}$$

iii) $a=0,\ b>0$: $\langle 0,x \rangle \le b$ valid for all $x \in P$

$$\longrightarrow \{ x \in P \mid \underbrace{\langle 0,x \rangle}_{=0} = \underbrace{b}_{>0} \} = \emptyset \text{ is a face}$$

$\mathcal{F}(P) := \{ \text{faces of } P \}$ ... set of faces

Properties of faces

Ex: try prove some (most proper-ties are "obvious" but not always easy to prove)

- faces are polytopes
  - trivial for $f = P$ or $f = \emptyset$
  - if $f = P \cap \partial H \longrightarrow f = \bigcap \mathcal{H} \cap H \cap \overline{H}$

$\longrightarrow$ faces have well-defined dimension

$$\dim f := \dim \operatorname{aff}(f)$$

dim	name
-1	$\emptyset$ = "nullity"
0	vertex
1	edge
2	"face"
3	cell
$\vdots$	$\vdots$
δ	δ-faces
$\vdots$	$\vdots$
$d-2$	ridge
$d-1$	facet
d	P itself

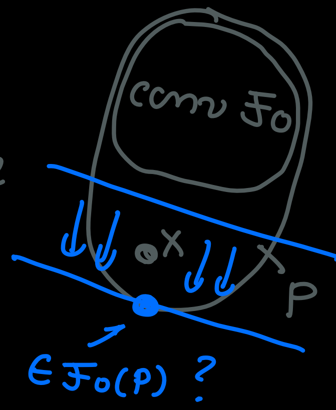
proper faces

$\mathcal{F}_\delta(P) := \{ \delta\text{-faces of } P \}$

- $P = \operatorname{conv} \mathcal{F}_0(P)$ (Krein - Milman Theorem) much more general

  Idea: - $\operatorname{conv} \mathcal{F}_0(P) \subseteq \operatorname{conv} P = P$

  - if $x \in P$ but $x \notin \operatorname{conv} \mathcal{F}_0(P)$, try separating with a hyperplane



- $P = \bigcap_{F} H(F)$ ← facet-defining halfspace

  (∗) F ← facets of P

(∗) really hard to show at this moment (see later)

⟶ $\emptyset \neq P$: P has a vertex

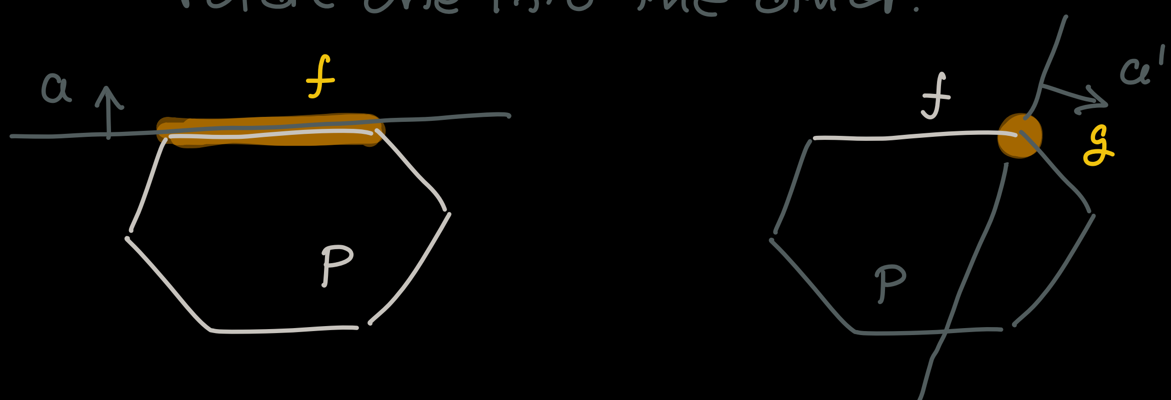
⟶ $\emptyset \neq P$: P has a facet (∗)

- "faces of faces are faces"

$$f \in \mathcal{F}(P) \longrightarrow \mathcal{F}(f) \subseteq \mathcal{F}(P)$$

Idea: $f \in \mathcal{F}(P)$, $g \in \mathcal{F}(f)$

take face-defining hyperplanes and rotate one into the other.



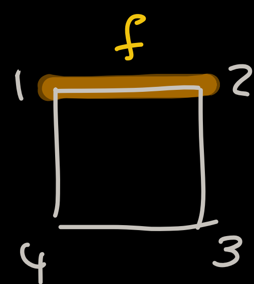
try $a + \varepsilon a'$ as normal vector

→ but what value for ε is suitable?

- more precisely $\mathcal{F}(f) = \{ g \in \mathcal{F}(P) \mid g \subseteq f \}$

- every face of P is completely determined by the vertices of P that it contains

$\longrightarrow$ P has only finitely many faces !



$f \simeq \{1,2\}$

<u>Example</u>

- in a d-simplex every subset of vertices defines a face

  $\longrightarrow$ 2^{d+1} faces

- this is the minimal amount possible in $\dim = d$

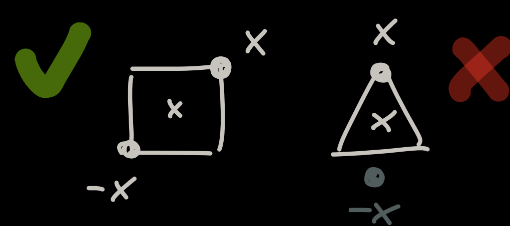
<u>Ex</u>: show that d-cube has 3^d faces

<u>Open</u>: (Kalai's 3^d-conjecture)

The d-cube has the minimal amount of faces of every <u>centrally symmetric</u> d-poly.

$P = -P$

"origin symmetric"



2.4 The face lattice

- $\mathcal{F}(P)$ is partially ordered by inclusion

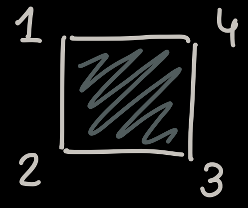
partial order : 1) reflexive $\quad f \subseteq f$

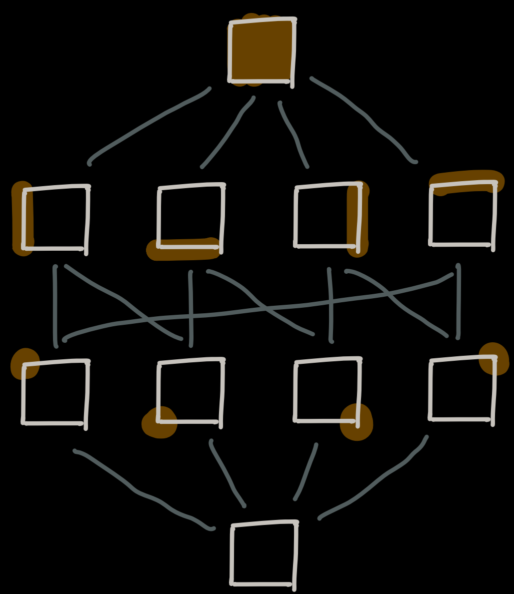
2) antisymmetric $\quad f \subseteq g \wedge g \subseteq f \longrightarrow f = g$

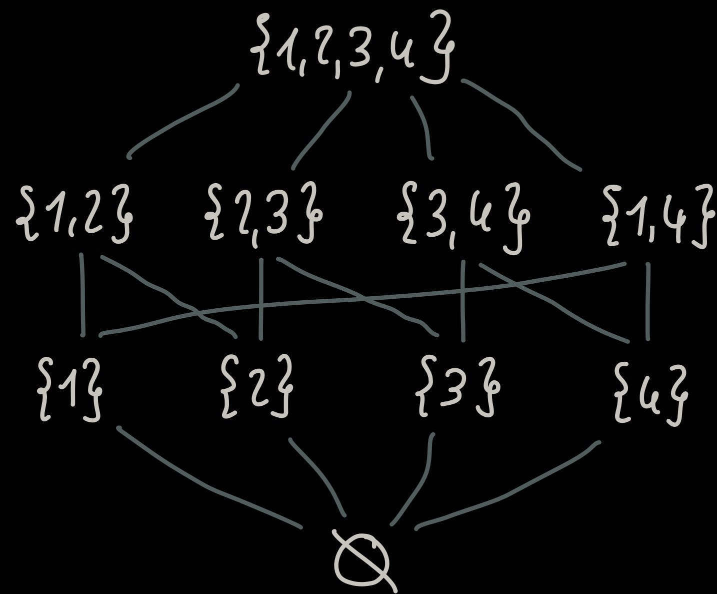
3) transitive $\quad f \subseteq g \subseteq h \longrightarrow f \subseteq h$

$\longrightarrow$ $(\mathcal{F}(P), \subseteq)$ is a partially ordered set (poset)

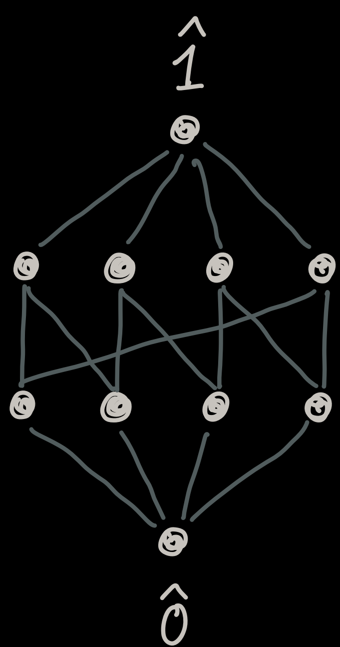
Example : square

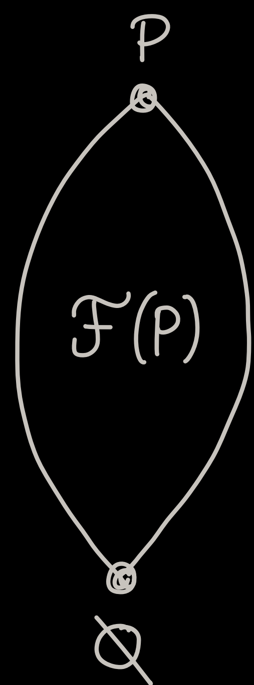




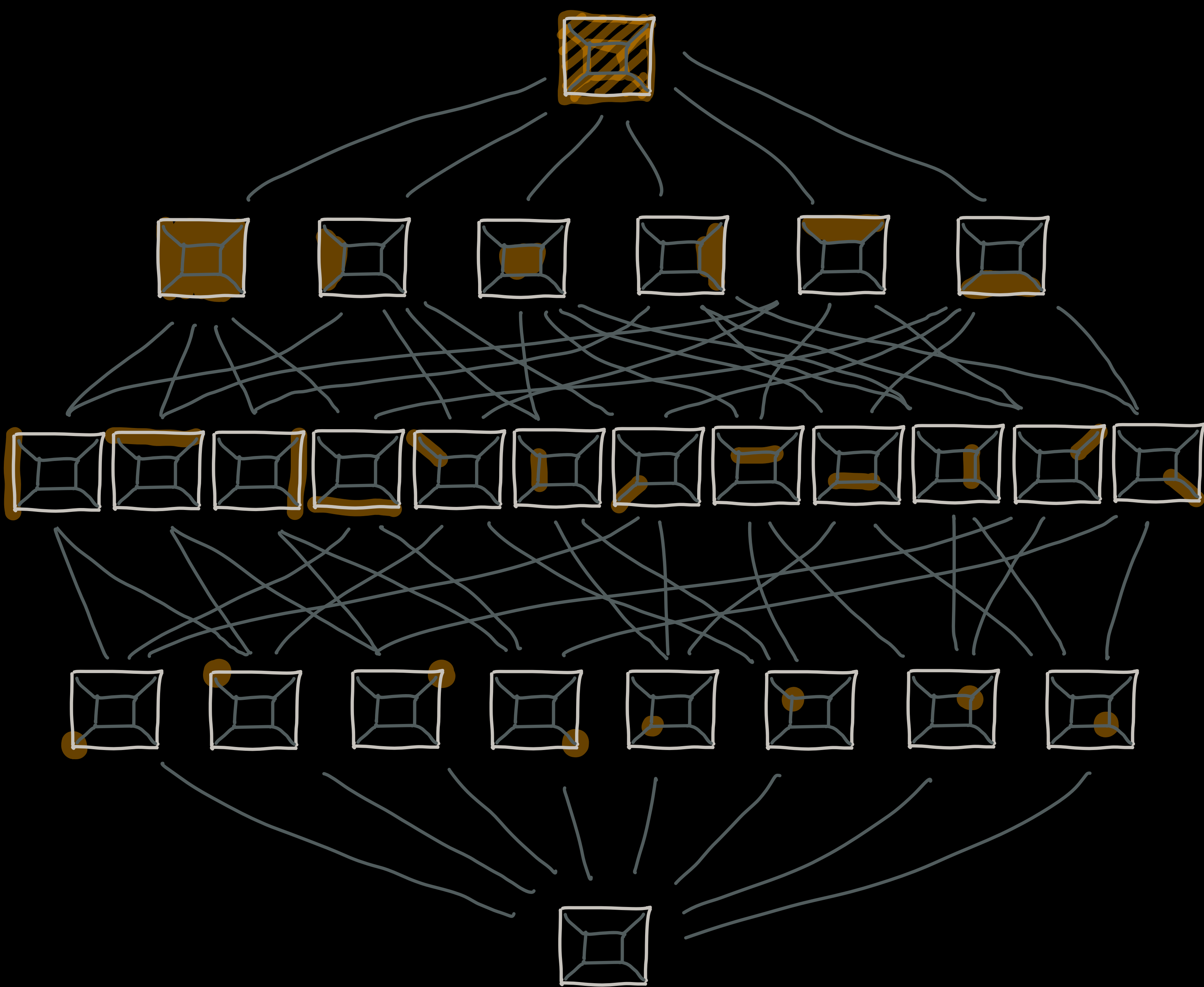


Hasse diagrams





Example: cube

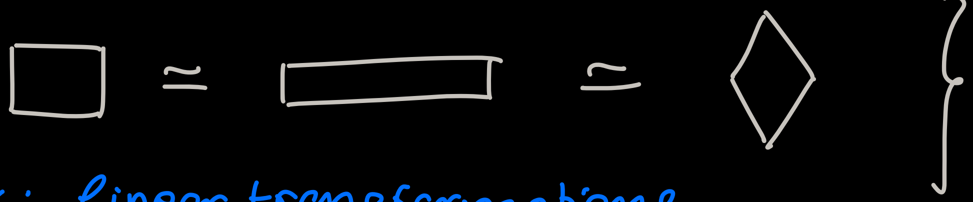


Def: P and Q are called combinatorially equivalent or of the same combinatorical type if

$$\mathcal{F}(P) \simeq \mathcal{F}(Q) \quad \text{isomorphic as posets}$$

face poset isomorphism

$$\left\{ \begin{array}{l} \varphi : \mathcal{F}(P) \longrightarrow \mathcal{F}(Q) \text{ bijective} \\ f \leq g \longrightarrow \varphi(f) \leq \varphi(g) \end{array} \right.$$



realizations of the same combinatorial type

Ex: linear transformations and translations preserve the combinatorial type.

Idea: face-defining hyperplanes are transformed as well.

- $\mathcal{F}(P)$ is more than just a poset

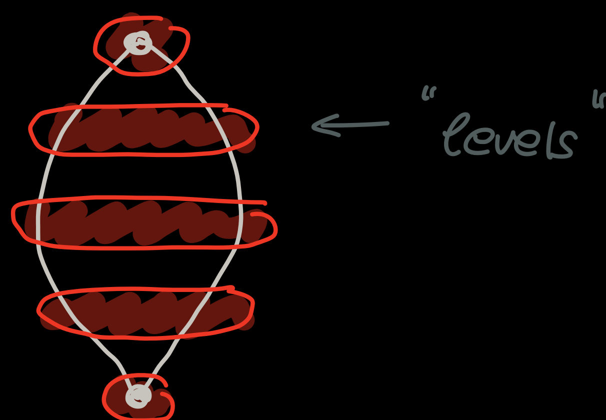
  $\longrightarrow$ it is a "complete graded lattice"

- complete means:

  there is a unique $\overset{P}{\text{top}}$ and $\overset{\emptyset}{\text{bottom}}$ element

- graded means:

  the elements are "sorted in levels" (*)



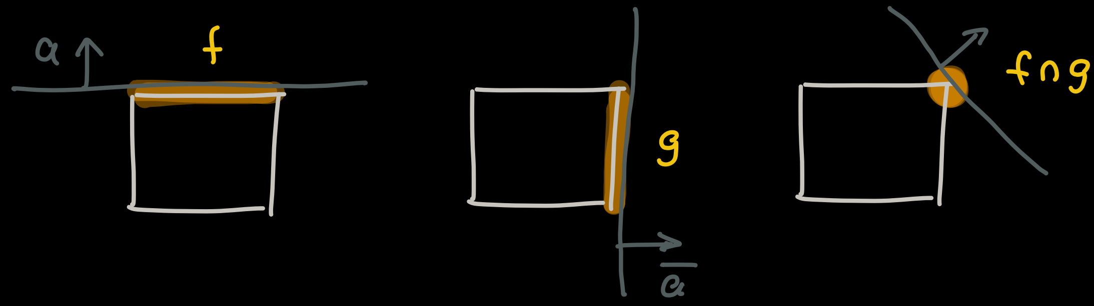
= every "path" from top to bottom has length $d+1$

- lattice means:

  for $f, g \in \mathcal{F}(P)$ exists a max and min

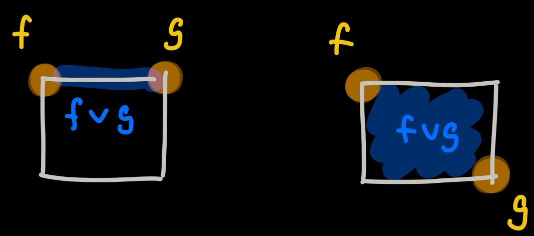
  (a lattice is a special order structure; not to be confused with lattices such as $\mathbb{Z}$)

– min : intersection of faces is a face



Idea: use $a + \bar{a}$

– max : there exists a unique minimal face that contains both f and g.



- Algorithmic considerations :

  Given a lattice $\mathcal{L}$, how hard is it to tell whether it is the face lattice of some polytope ?

→ NP-hard (for dim ≥ 4)

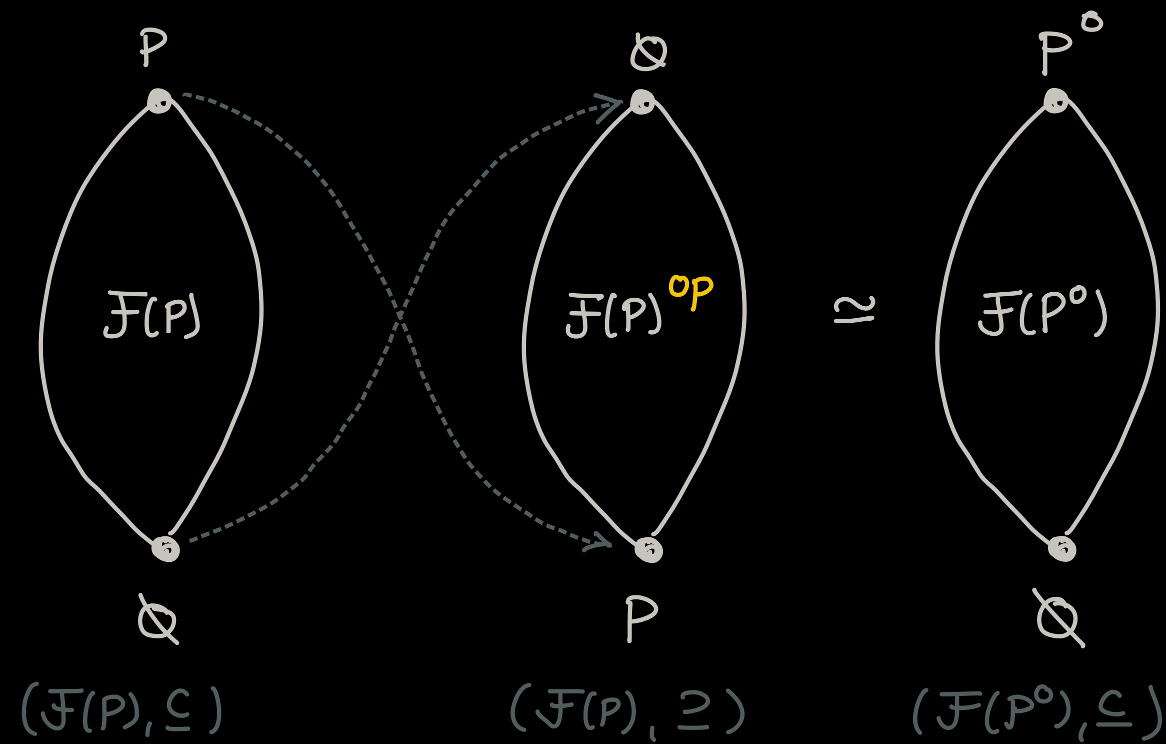
Open : is it NP-complete ?
is it coNP-complete ?

probably neither

NOTE : it is decidable !!
(this was wrongly stated in the lecture)

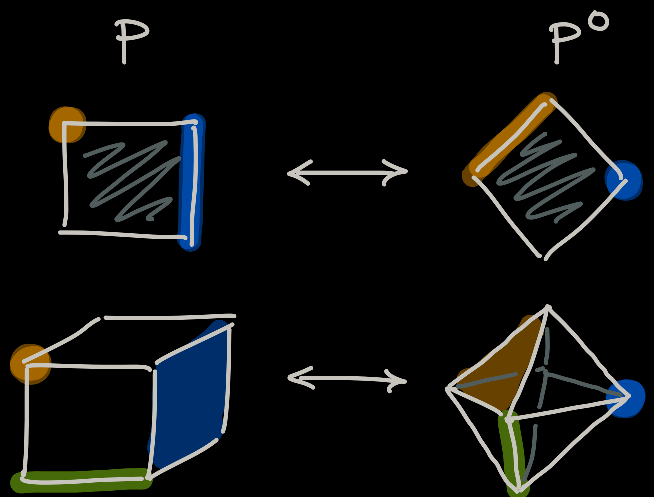
2.5. Duality and the flipped face lattice

" polar duality flips the face lattice "



- every face $f \in \mathcal{F}(P)$ has a dual face in $\mathcal{F}(P^\circ)$

Example :



vertex $\leftrightarrow$ edge

vertex $\leftrightarrow$ 2-face
edge $\leftrightarrow$ edge

δ-face $\leftrightarrow$ $(d-1-\delta)$-face

dual face

Def: $f \in \mathcal{F}(P)$ $\quad f^\Delta := \{ y \in P^\circ \mid \langle x, y \rangle = 1 \ \forall x \in f \}$

not need $\quad = \{ y \in P^\circ \mid \langle x_i, y \rangle = 1 \ \forall x_i \in \mathcal{F}_0(f) \}$

Thm:

$\varphi : \mathcal{F}(P) \longrightarrow \mathcal{F}(P^\circ)$

(i) f^Δ is a face of P°

(ii) $f^{\Delta\circ} = f$

(iii) $f \subset g \longrightarrow g^\Delta \subset f^\Delta$

(iv) $\dim f^\Delta = d-1-\dim f$

$\varphi : f \mapsto f^\Delta$ is well-defined

φ is a bijection

φ is order reversing

$$\underbrace{\qquad\qquad\qquad\qquad}_{\mathcal{F}(P)^{op} \simeq \mathcal{F}(P^\circ)}$$

Proof sketch: (not included in the lecture)

(i) $f^\Delta = \bigcap_{x \in \mathcal{F}_0(P)} (P^\circ \cap \partial H(x,1))$ is intersection of faces, hence a face

(ii) similar computation to $P^{\circ\circ} = P$

(iii) trivial

(iv) (*) $\square$

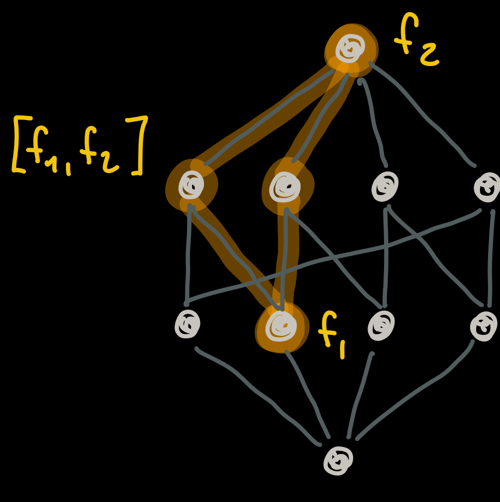
- a polytope is called (combinatorially) self-dual if $\mathcal{F}(P) \simeq \mathcal{F}(P)^{op}$

Open: If P is self-dual, is it combinatorially equivalent to a self-polar polytope?

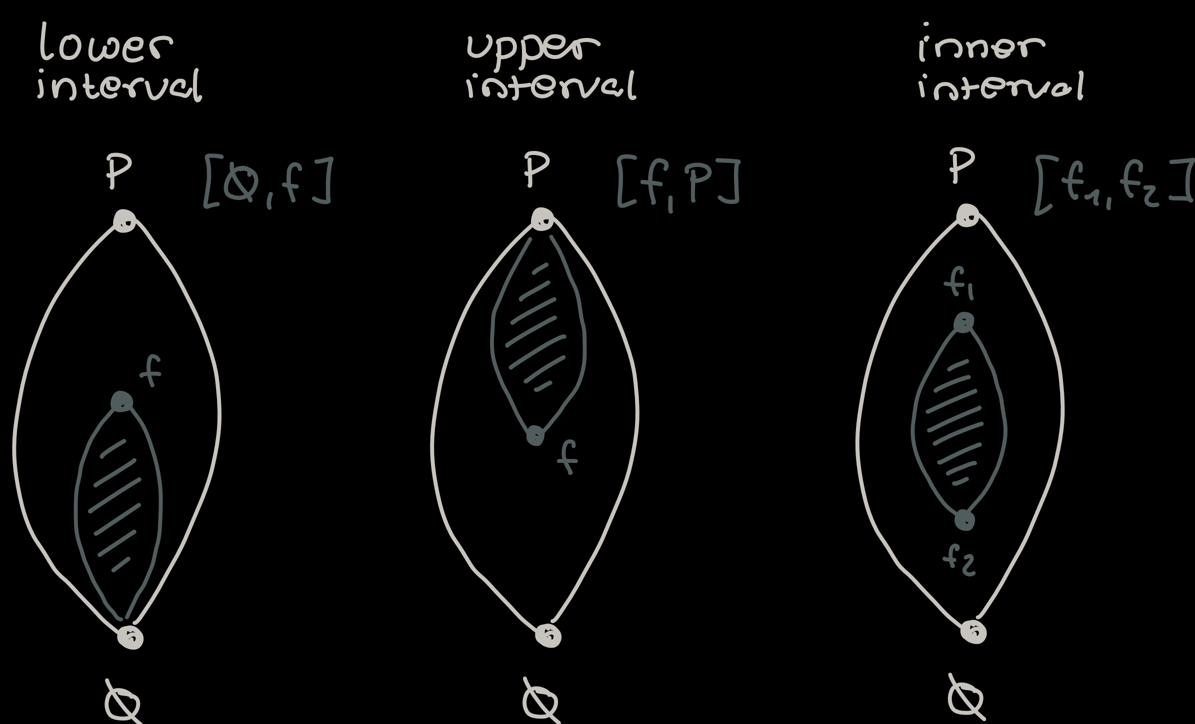
2.6. Intervals and vertex figures

Def: for $f_1, f_2 \in \mathcal{F}(P)$ the interval is

$$[f_1, f_2] := \{ g \in \mathcal{F}(P) \mid f_1 \subseteq g \subseteq f_2 \}$$



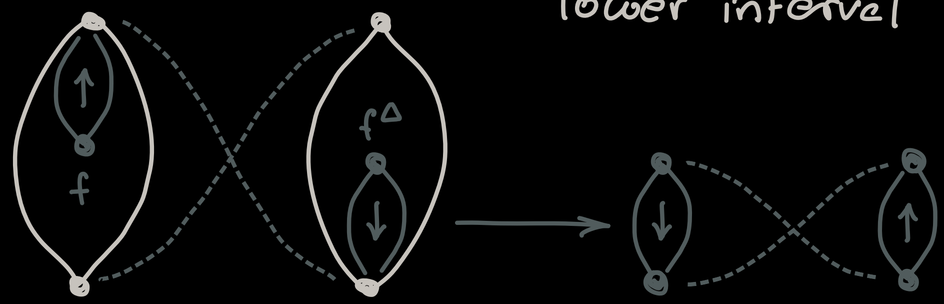
Question: Are intervals in face lattices again face lattices of some polytopes?



Yes! In all three cases

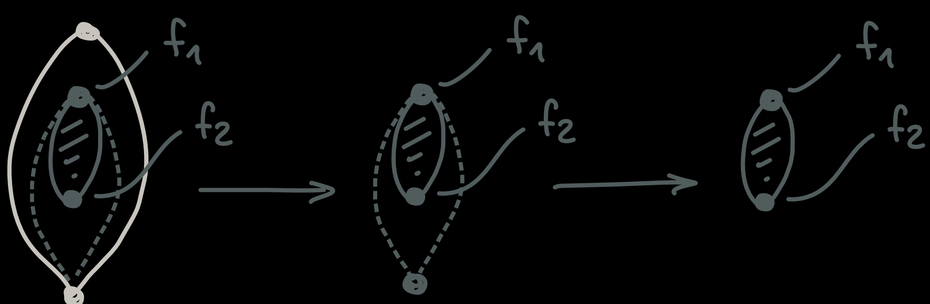
- <u>lower intervals:</u> $[\emptyset, f] \simeq \mathcal{F}(f)$ (easy to see)

- <u>upper intervals:</u> take opposite lattice and then a lower interval



$\mathcal{F}(P)$ $\quad$ $\mathcal{F}(P^o)$ $\quad$ $\mathcal{F}(f^\Delta)$ $\quad$ $\mathcal{F}((f^\Delta)^o) \simeq [f, P]$

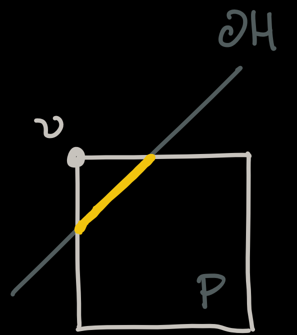
- <u>inner intervals:</u> take lower interval and then upper interval

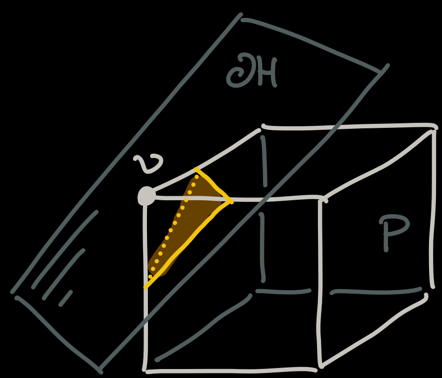


For a vertex $v \in \mathcal{F}_0(P)$ there exists a nice geometric interpretation for the upper interval

$$[v,P] \simeq \mathcal{F}(P/v)$$

vertex figure





$P/v \simeq$ 

$P/v \simeq$ 

∂H ... hyperplane that separates v from $\mathcal{F}_0(P) \setminus \{v\}$

$$P/v := P \cap \partial H$$

<u>NOTE</u>: depends on choice of ∂H

BUT combinatorics is independent of ∂H.

<u>Idea</u>:

$$\mathcal{F}(P) \ni f \longmapsto f \cap \partial H \in \mathcal{F}(P/v)$$

$$\mathcal{F}(P) \ni P \cap \operatorname{aff}(f' \cup \{v\}) \longleftarrow f' \in \mathcal{F}(P/v)$$

Using that P/v is a $(d-1)$-polytope one can now finally prove all of $(*)$ Ex: try it!

- face lattice is graded = all maximal chains have length $d-1$
- dual to δ-face has dimension $d-1-\delta$
- each polytope has a facet (Idea: take dual of vertex)
- each face is intersection of some facets